Announcements

1) Candidate talk tomorrow
   3-4 2090 CB
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2) HW 4 up sometime
   later today

Definition: (metric space) Let X be a set. A **metric** on X is a function

$$d : X \times X \to [0, \infty)$$

Satisfying

1) $d(x,x) = 0 \quad \forall\, x \in X$ and if $d(x,y) = 0$, then $x = y \quad \forall\, x,y \in X$

2) $d(x,y) = d(y,x) \quad \forall\, x,y \in X$

3) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall\, x,y,z \in X$ (triangle inequality)

A metric is also called a distance function.

A set X endowed with a metric d is called a metric space.

Examples:

1) $\mathbb{R}$ with $d(x,y) = |x-y|$

is a metric space.

1st 2 properties of a metric are clear, 3rd we've already proved

The same metric will also work for

$X = \mathbb{Q}, \mathbb{Z}, [0,1],$

almost any subset of $\mathbb{R}$.

2) $X = \mathbb{R}^2$,

$$d_p((x_1, x_2), (y_1, y_2)) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}$$

for $1 \leq p < \infty$.

This gives a family of (equivalent) metrics on $\mathbb{R}^2$. With $p = 2$, this is the Euclidean metric.

The only property to worry about is the triangle inequality - but it does hold!

3) X = any nonempty set.

Define

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

$\forall \; x, y \in X$.

Claim: this is a metric

Check properties

i) $d(x,y)=0$ if and only if $x=y$

immediate by construction since $d(x,y)=1$ if $x\neq y$.

ii) $d(x,y)=d(y,x)$

If $x=y$, then $d(x,y)=d(x,x)=d(y,x)$
$=0$

If $x\neq y$, then

$$d(x,y)=1=d(y,x).$$

iii) $d(x,y)\leq d(x,z)+d(z,y)$

If $x=y$, $d(x,y)=0$,

so automatically

$d(x,y)=0 \leq d(x,z)+d(z,y)$

for any choice of z since

$d: X \to [0,\infty)$.

If $x \neq y$, $d(x,y)=1$

Then $d(x,z)+d(z,y) \geq 1$

since if $d(x,z)=0$, then

$z=x$ and $y \neq x \Rightarrow d(z,y)=1$,

and similarly if $d(z,y)=0$,

then $d(z,x)=1$. $\square$

Definition: (convergence / Cauchy)

Let X be a metric space with metric d. A sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ is said to **converge** if $\exists\, x \in X$ such that $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$\boxed{d(x_n, x) < \varepsilon}$$

$$\forall\, n \geq N$$

A sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ is said to be Cauchy if $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

$$\forall\; n, m \geq N.$$

Note: there's no mention of a limit, just that the terms "become close" as you go further in the sequence.

Proposition: Let X be a metric space with metric d. Then if $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$, then $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X.

proof: Let $\varepsilon > 0$. We want

$$d(x_n, x_m) < \varepsilon.$$

To get this, note that

$\forall\, n, m \in \mathbb{N},$

$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$

by the triangle inequality.

If we want this quantity to be less than ε, choose $N \in \mathbb{N}$ so that

$d(x_n, x) < \varepsilon/2$ for all $n \geq N$.

Then for all $n, m \geq N$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \\ &= d(x_n, x) + d(x_m, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square \end{aligned}$$

Example 1 Let $X = \mathbb{Q}$ and $d(x,y) = |x-y|$.

Let $a_1 = 1$, $a_n = 1 + \frac{1}{1+a_{n-1}}$.

Then $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

You showed on your homework that the even terms and the odd terms both converge.

In fact, both subsequences converge to $\sqrt{2} \notin \mathbb{Q}$.

Since $a_n \to \sqrt{2}$, $(a_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, hence Cauchy in $\mathbb{Q}$ (by the previous proposition)

This shows that $\mathbb{Q}$ contains Cauchy sequences which do not converge to any rational number

What about the converse?

Given a metric space, must every Cauchy sequence in that space converge to an element in that space?

False for even familiar metric spaces.

If X is a metric space and every Cauchy sequence in X converges to an element of X, then we say X is complete.

Lemma: Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be Cauchy with the absolute value metric. Then $(x_n)_{n\in\mathbb{N}}$ is bounded.

proof: Let $\varepsilon = \frac{1}{5}$. Then $\exists N \in \mathbb{N}$, $|x_n - x_m| < \frac{1}{5}$ for all $n, m \geq N$.

If $n \geq N$,

$$|x_n| \leq |x_N - x_n| + |x_N|$$
$$< \frac{1}{5} + |x_N|$$

Let

$$M = \max\left\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + \frac{1}{5}\right\}$$

Then if $n \in \mathbb{N}$,

$$|x_n| \leq M.$$

$\square$

Theorem: If $X = \mathbb{R}$ with the metric $d(x,y) = |x-y|$, then $\mathbb{R}$ is complete.

proof next time